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COMMENT

Extension of IMH method to electrovac fields

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Abstract. An algebraic method due to Ihrig has been recently developed further for four-dimensional space–times by McIntosh and Halford to determine the metric tensor from the components of the Riemann tensor. This method has been extended to electrovac fields in this paper.

1. Introduction

Ihrig (1975) has given an algebraic method for finding the metric tensor $g_{\mu\nu}$ (up to a conformal factor) from the components of the Riemann tensor $R^{\mu}_{\ \nu\alpha\beta}$ in some coordinate frame from the identity

$$g_{\mu(\nu}R^{\mu}_{\ \lambda)\alpha\beta}=0. \tag{1}$$

This method has been developed further by McIntosh and Halford (1981) for four-dimensional space–times. We extend the Ihrig–McIntosh–Halford (IMH) technique here to $(4+1)$ dimensions so that it may be applied to electrovac fields. We present the method of computation in § 2 and work out two problems, one spherically symmetric and the other axially symmetric, in § 3.

2. Method of computation

Let us consider a vector space X spanned by fifteen orthonormal vectors $x_{\mu\nu}=x_{\nu\mu}$ having the inner product

$$(x_{\mu_1\nu_1}, x_{\mu_2\nu_2})=\delta_{\mu_1\mu_2}, \delta_{\nu_1\nu_2} \tag{2}$$

where the Greek indices range over 0, 1, 2, 3, 4.

In X we form as many linearly independent vectors v_α $(\alpha=1, 2, 3, \ldots, m; m\leqslant 14)$ as possible from the expression

$$\psi x_{\mu(\nu}R^{\mu}_{\ \lambda)\alpha\beta} \tag{3}$$

where ψ is an arbitrary scalar function. The lengths of these v_α vectors are not important, and so ψ is chosen to give v_α simple forms as linear combinations of the $x_{\mu\nu}$.

Next, we write down the most general vector ω in X which is orthogonal to the vectors v_α. ω may have more than one arbitrary function in it.

Finally, metric components are calculated from the inner product

$$kg_{\mu\nu} = (\omega x_{\mu\nu}). \tag{4}$$

3. Examples

3.1. Reissner–Nordstrom field

Let us consider the Reissner–Nordstrom metric in (4 + 1) dimensions as follows

$$ds^2 = e^{2\nu}dt^2 - e^{-2\nu}\,dR^2 - R^2[dx^2 + \sin^2 x(d\theta^2 + \sin^2\theta\,d\phi^2)] \tag{5}$$

where

$$e^{2\nu} = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} \tag{6}$$

t represents the time coordinate and R, x, θ and ϕ represent space coordinates. The usual form of the Reissner–Nordstrom metric is obtained with $x = \pi/2$.

The components of the Riemann tensor obtained are

$$\begin{aligned}
&R^0{}_{220} = R^0{}_{330} = R^0{}_{440} = R^2{}_{112} = R^3{}_{113} = R^4{}_{114} = e^{2\nu}\nu'/R \\
&R^1{}_{221} = e^{4\nu}\nu' R \qquad R^4{}_{004} = \frac{e^{4\nu}\nu'}{R^3\sin^2\theta\sin^2 x} \\
&R^1{}_{010} = e^{6\nu}(2\nu'^2 + \nu'') \qquad R^0{}_{110} = e^{2\nu}(2\nu'^2 + \nu'') \\
&R^2{}_{020} = e^{4\nu}\nu'/R^3 \qquad R^3{}_{030} = \frac{e^{4\nu}\nu'}{R^3\sin^2 x} \qquad R^1{}_{331} = e^{4\nu}R\nu'\sin^2 x \\
&R^3{}_{434} = \frac{\sin^2\theta(1 - e^{2\nu})}{R^2} \qquad R^4{}_{242} = \frac{(1 - e^{2\nu})}{R^2\sin^2\theta\sin^2 x} \\
&R^3{}_{232} = R^4{}_{343} = R^2{}_{424} = \frac{(1 - e^{2\nu})}{R^2} \\
&R^1{}_{441} = e^{4\nu}R\nu'\sin^2\theta\sin^2 x \qquad R^2{}_{332} = \frac{\sin^2 x(1 - e^{2\nu})}{R^2}.
\end{aligned} \tag{7}$$

The linearly independent vectors v_α found with the help of Riemann tensor components are

$$\begin{aligned}
&v_1 = x_{00}\,e^{-2\nu} + x_{11}\,e^{2\nu} \qquad v_2 = x_{00}\,e^{-2\nu} + x_{22}/R^2 \\
&v_3 = x_{00}\,e^{-2\nu} + x_{33}/R^2\sin^2 x \qquad v_4 = x_{00}\,e^{-2\nu} + x_{44}/R^2\sin^2 x\sin^2\theta \\
&v_5 = x_{01} \qquad v_6 = x_{02} \qquad v_7 = x_{03} \\
&v_8 = x_{04} \qquad v_9 = x_{12} \qquad v_{10} = x_{13} \\
&v_{11} = x_{14} \qquad v_{12} = x_{23} \qquad v_{13} = x_{24} \\
&v_{14} = x_{34}.
\end{aligned} \tag{8}$$

The value of ω which is orthogonal to all vectors is given by

$$\omega = e^{2\nu}x_{00} - e^{-2\nu}x_{11} - R^2[x_{22} + \sin^2 x(x_{33} + \sin^2\theta\,x_{44})]. \tag{9}$$

Therefore, we may have the metric as

$$ds^2 = \phi\{e^{2\nu}\,dt^2 - e^{-2\nu}\,dR^2 - R^2[dx^2 + \sin^2 x(d\theta^2 + \sin^2\theta\,d\phi^2)]\}. \tag{10}$$

where ϕ is an arbitrary scalar.

3.2. Schwarzschild–Ernst field

The Schwarzschild–Ernst (Ernst 1976) metric in $(4+1)$ dimensions is

$$ds^2 = \lambda^2 A\,dt^2 - \lambda^2 A^{-1}\,dr^2 - \lambda^2 r^2[dx^2 + \sin^2 x(d\theta^2 + \lambda^{-4}\sin^2\theta\,d\phi^2)] \tag{11}$$

where

$$\lambda = 1 + \tfrac{1}{4}B_0^2 r^2 \sin^2\theta \qquad \text{and} \qquad A = 1 - 2m/r. \tag{12}$$

Here B_0 is a parameter related to the magnetic field in which a Schwarzschild object is immersed.

The non-zero components of the Riemann tensor are

$$R^0{}_{110} = \frac{B_0^4 r^2 \sin^2\theta\cos^2\theta}{4\lambda^2 A\sin^2 x} + \frac{B_0^4 r^2\sin^4\theta}{4\lambda^2} + \frac{2m}{Ar^3} - \frac{mB_0^2\sin^2\theta}{\lambda A r} - \frac{B_0^2\sin^2\theta}{2\lambda}$$

$$R^0{}_{103} = \frac{3B_0^2\sin\theta\cos\theta}{2\lambda A^{1/2}\sin x} \qquad R^1{}_{003} = \frac{3A^{3/2}B_0^2\sin\theta\cos\theta}{2\lambda\sin x}$$

$$R^1{}_{010} = \frac{A^2 B_0^4 r^2\sin^4\theta}{4\lambda^2} + \frac{AB_0^4 r^2\sin^2\theta\cos^2\theta}{4\lambda^2\sin^2 x} + \frac{2Am}{r^3} - \frac{A^2B_0^2\sin^2\theta}{2\lambda} - \frac{mAB_0^2\sin^2\theta}{\lambda r}$$

$$R^0{}_{220} = \frac{AB_0^2 r^2\sin^2\theta}{2\lambda} + \frac{AB_0^4 r^4\sin^4\theta}{4\lambda^2} + \frac{m}{r} + \frac{mB_0^2 r\sin^2\theta}{2\lambda} + \frac{B_0^4 r^4\sin^2\theta\cos^2\theta}{4\lambda^2\sin^2 x}$$

$$R^2{}_{020} = \frac{A^2B_0^2\sin^2\theta}{2\lambda} + \frac{A^2B_0^4 r^2\sin^4\theta}{4\lambda^2} + \frac{Am}{r^3} + \frac{AmB_0^2\sin^2\theta}{2\lambda r} + \frac{AB_0^4 r^2\sin^2\theta\cos^2\theta}{4\lambda^2\sin^2 x}$$

$$R^0{}_{203} = \frac{B_0^2 r^2\sin\theta\cos\theta\cot x}{2\lambda\sin x} \qquad R^2{}_{003} = \frac{AB_0^2\sin\theta\cos\theta\cot x}{2\lambda\sin x}$$

$$R^0{}_{404} = \frac{AB_0^4 r^4\sin^6\theta\sin^2 x}{4\lambda^6} + \frac{B_0^4 r^4\sin^4\theta\cos^2\theta}{4\lambda^6} + \frac{mB_0^2 r\sin^4\theta\sin^2 x}{2\lambda^5}$$
$$- \frac{AB_0^2 r^2\sin^4\theta\sin^2 x}{2\lambda^5} - \frac{m\sin^2\theta\sin^2 x}{\lambda^4 r} - \frac{B_0^2 r^2\sin^2\theta\cos^2\theta}{2\lambda^5}$$

$$R^4{}_{004} = \frac{A^2B_0^4 r^2\sin^4\theta}{4\lambda^2} + \frac{AB_0^4 r^2\sin^2\theta\cos^2\theta}{4\lambda^2\sin^2 x} + \frac{AB_0^2 m\sin^2\theta}{2\lambda r}$$
$$- \frac{A^2B_0^2\sin^2\theta}{2\lambda} - \frac{mA}{r^3} - \frac{AB_0^2\cos^2\theta}{2\lambda\sin^2 x}$$

$$R^0{}_{301} = \frac{B_0^2 r^2\sin\theta\cos\theta\sin x}{2\lambda A^{1/2}} \qquad R^3{}_{001} = \frac{B_0^2 A^{1/2}\sin\theta\cos\theta}{2\lambda\sin x}$$

$$R^0{}_{320} = \frac{B_0^2 r^2\sin\theta\cos\theta\cos x}{2\lambda} \qquad R^3{}_{020} = \frac{AB_0^2\sin\theta\cos\theta\cos x}{2\lambda\sin^2 x}$$

$$R^0{}_{330}=\frac{B_0^4r^4\sin^2\theta\cos^2\theta}{4\lambda^2}+\frac{AB_0^4r^4\sin^4\theta\sin^2x}{4\lambda^2}+\frac{AB_0^2r^2\sin^2\theta\sin^2x}{2\lambda}$$
$$+\frac{mB_0^2r\sin^2\theta\sin^2x}{2\lambda}+\frac{m\sin^2x}{r}-\frac{B_0^2r^2(\cos^2\theta-\sin^2\theta)}{2\lambda}$$

$$R^3{}_{030}=\frac{Am}{r^3}-\frac{AB_0^2(\cos^2\theta-\sin^2\theta)}{2\lambda\sin^2x}+\frac{AB_0^4r^2\sin^2\theta\cos^2\theta}{4\lambda^2\sin^2x}+\frac{A^2B_0^2\sin^2\theta}{2\lambda}$$
$$+\frac{A^2B_0^4r^2\sin^4\theta}{4\lambda^2}+\frac{AB_0^2m\sin^2\theta}{2\lambda r}$$

$$R^1{}_{212}=\frac{AB_0^4r^4\sin^4\theta}{4\lambda^2}+\frac{B_0^4r^4\sin^2\theta\cos^2\theta}{4\lambda^2\sin^2x}-\frac{mB_0^2r\sin^2\theta}{2\lambda}-\frac{m}{r}-\frac{AB_0^2r^2\sin^2\theta}{\lambda}$$

$$R^2{}_{112}=\frac{m}{Ar^3}+\frac{B_0^2\sin^2\theta}{\lambda}-\frac{B_0^4r^2\sin^4\theta}{4\lambda^2}-\frac{B_0^4r^2\sin^2\theta\cos^2\theta}{4\lambda^2A\sin^2x}+\frac{B_0^2m\sin^2\theta}{2\lambda Ar}$$

$$R^1{}_{223}=\frac{3A^{1/2}B_0^2r^2\sin\theta\cos\theta}{2\lambda\sin x}\qquad R^2{}_{132}=\frac{3B_0^2\sin\theta\cos\theta}{2\lambda A^{1/2}\sin x}$$

$$R^1{}_{231}=\frac{B_0^2r^2\sin\theta\cos\theta\cos x}{2\lambda\sin^2x}\qquad R^2{}_{113}=\frac{B_0^2\sin\theta\cos\theta\cos x}{2\lambda A\sin^2x}$$

$$R^1{}_{313}=\frac{B_0^4r^4\sin^2\theta\cos^2\theta}{4\lambda^2}+\frac{AB_0^4r^4\sin^4\theta\sin^2x}{4\lambda^2}-\frac{m\sin^2x}{r}-\frac{AB_0^2r^2\sin^2\theta\sin^2x}{\lambda}$$
$$-\frac{B_0^2mr\sin^2\theta\sin^2x}{2\lambda}-\frac{B_0^2r^2(\cos^2\theta-\sin^2\theta)}{2\lambda}$$

$$R^3{}_{113}=\frac{m}{Ar^3}+\frac{B_0^2\sin^2\theta}{\lambda}-\frac{B_0^4r^2\sin^4\theta}{4\lambda^2}+\frac{B_0^2m\sin^2\theta}{2\lambda Ar}$$
$$+\frac{B_0^2(\cos^2\theta-\sin^2\theta)}{2\lambda A\sin^2x}-\frac{B_0^4r^2\sin^2\theta\cos^2\theta}{4\lambda^2A\sin^2x}$$

$$R^1{}_{312}=\frac{B_0^2r^2\sin\theta\cos\theta\cos x}{2\lambda}\qquad R^3{}_{112}=\frac{-B_0^2\sin\theta\cos\theta\cos x}{2\lambda A\sin^2x}$$

$$R^1{}_{332}=\frac{A^{1/2}\sin x\cos x}{\lambda}(2+\tfrac{3}{2}B_0^2r^2\sin^2\theta)\qquad R^3{}_{123}=\frac{\cot x}{A^{1/2}}\left(\frac{B_0^2\sin^2\theta}{\lambda}+\frac{2}{r^2}\right)$$

$$R^1{}_{414}=\frac{r^2\sin^2\theta\sin^2x}{\lambda^4}\left(\frac{B_0^2m\sin^2\theta}{2\lambda r}+\frac{3AB_0^2\sin^2\theta}{2\lambda}-\frac{3AB_0^4r^2\sin^4\theta}{4\lambda^2}-\frac{m}{r^3}\right.$$
$$\left.-\frac{B_0^4r^2\sin^2\theta\cos^2\theta}{4\lambda^2\sin^2x}+\frac{B_0^2\cos^2\theta}{2\lambda\sin^2x}\right)$$

$$R^4{}_{114}=\frac{B_0^4r^2\sin^2\theta\cos^2\theta}{4\lambda^2A\sin^2x}+\frac{3B_0^4r^2\sin^4\theta}{4\lambda^2}+\frac{m}{Ar^3}-\frac{B_0^2\sin^2\theta}{2\lambda}$$
$$-\frac{B_0^2m\sin^2\theta}{2\lambda Ar}-\frac{B_0^2\cos^2\theta}{2\lambda A\sin^2x}$$

$$R^1{}_{442} = \frac{2A^{1/2}\sin^2\theta \sin x \cos x}{\lambda^4} \qquad R^4{}_{124} = \frac{2\cot x}{A^{1/2}r^2}$$

$$R^1{}_{434} = \frac{A^{1/2}r^2\sin^2\theta \sin x}{\lambda^4}\left(\frac{5B_0^2 \sin\theta\cos\theta}{2\lambda} - \frac{2\cot\theta}{r^2} - \frac{B_0^4 r^2 \sin^3\theta\cos\theta}{2\lambda^2}\right)$$

$$R^4{}_{134} = \frac{\cos\theta}{A^{1/2}\sin x}\left(\frac{2}{\sin\theta} - \frac{5B_0^2\sin\theta}{2\lambda} + \frac{B_0^4 r^2\sin^3\theta}{2\lambda^2}\right)$$

$$R^2{}_{312} = -R^4{}_{314} = \frac{B_0^2 A^{1/2} r^2 \sin\theta\cos\theta\sin x}{2\lambda}\left(3 - \frac{B_0^2 r^2\sin^2\theta}{\lambda}\right)$$

$$R^3{}_{212} = \frac{B_0^2 A^{1/2} r^2 \sin\theta\cos\theta}{2\lambda\sin x}\left(\frac{B_0^2 r^2\sin^2\theta}{\lambda} - 3\right)$$

$$R^2{}_{323} = (A+1)\sin^2 x + \frac{AB_0^4 r^4 \sin^4\theta\sin^2 x}{4\lambda^2} - \frac{B_0^2 r^2(\cos^2\theta - \sin^2\theta)}{2\lambda} + \frac{B_0^4 r^4\sin^2\theta\cos^2\theta}{4\lambda^2} + \frac{AB_0^2 r^2\sin^2\theta\sin^2 x}{\lambda}$$

$$R^3{}_{223} = \frac{B_0^2 r^2(\cos^2\theta - \sin^2\theta)}{2\lambda\sin^2 x} - \frac{B_0^4 r^4\sin^2\theta\cos^2\theta}{4\lambda^2\sin^2 x} - \frac{AB_0^4 r^4\sin^4\theta}{4\lambda^2} - (A+1) - \frac{AB_0^2 r^2\sin^2\theta}{\lambda}$$

$$R^2{}_{414} = \frac{B_0^2 A^{1/2} r^2 \sin^4\theta\sin x\cos x}{\lambda^5} \qquad R^4{}_{214} = -\frac{B_0^2 A^{1/2} r^2\sin^2\theta\cot x}{\lambda}$$

$$R^2{}_{434} = \frac{\sin\theta\cos\theta\cos x}{\lambda^5}(B_0^2 r^2\sin^2\theta - 2)$$

$$R^4{}_{234} = \frac{2\cot\theta\cos x}{\sin^2 x} - \frac{3B_0^2 r^2\sin\theta\cos\theta\cos x}{2\lambda\sin^2 x}$$

$$R^2{}_{424} = \frac{(A+1)\sin^2\theta\sin^2 x}{\lambda^4} - \frac{AB_0^4 r^4\sin^6\theta\sin^2 x}{4\lambda^6} - \frac{B_0^4 r^4\sin^4\theta\cos^2\theta}{4\lambda^6} + \frac{B_0^2 r^2\sin^2\theta\cos^2\theta}{2\lambda^5}$$

$$R^4{}_{224} = \frac{AB_0^4 r^4\sin^4\theta}{4\lambda^2} + \frac{B_0^4 r^4\sin^2\theta\cos^2\theta}{4\lambda^2\sin^2 x} - \frac{B_0^2 r^2\cos^2\theta}{2\lambda\sin^2 x} - (A+1)$$

$$R^3{}_{442} = \frac{B_0^2 r^2\sin^3\theta\cos\theta\cos x}{2\lambda^5} \qquad R^4{}_{324} = \frac{B_0^2 r^2\sin\theta\cos\theta\cos x}{2\lambda}$$

$$R^3{}_{414} = \frac{B_0^2 A^{1/2} r^2\sin^3\theta\cos\theta\sin x}{2\lambda^5}\left(3 - \frac{B_0^2 r^2\sin^2\theta}{\lambda}\right)$$

$$R^3{}_{434}=\frac{(A-1)\sin^2\theta\sin^2 x}{\lambda^4}+\frac{2\sin^2\theta}{\lambda^4}+\frac{B_0^2r^2(4\cos^2\theta-\sin^2\theta)\sin^2\theta}{2\lambda^5}$$
$$-\frac{3B_0^4r^4\sin^4\theta\cos^2\theta}{4\lambda^6}-\frac{AB_0^4r^4\sin^6\theta\sin^2 x}{4\lambda^6}$$

$$R^4{}_{334}=(1-A)\sin^2 x-\frac{B_0^2r^2(4\cos^2\theta-\sin^2\theta)}{2\lambda}$$
$$+\frac{3B_0^4r^4\sin^2\theta\cos^2\theta}{4\lambda^2}+\frac{AB_0^4r^4\sin^4\theta\sin^2 x}{4\lambda^2}-2$$

$$R^\mu{}_{\nu\alpha\beta}=-R^\mu{}_{\nu\beta\alpha}. \tag{13}$$

It is found that the linearly independent vectors v_α are

$$v_1=x_{00}\lambda^2A^{-1}+\lambda^2Ax_{11}+\lambda^{-2}Ax_{13}R^3{}_{001}/R^0{}_{101}$$
$$v_2=x_{00}\lambda^2r^2+\lambda^2Ax_{22}+\lambda^{-2}r^{-2}x_{23}R^3{}_{020}/R^0{}_{202}$$
$$v_3=\lambda^2r^2\sin^2 x\,x_{00}+\lambda^2Ax_{33}+\frac{1}{\lambda^2r^2\sin^2 x}(x_{13}R^1{}_{003}/R^0{}_{303}+x_{23}R^2{}_{003}/R^0{}_{303})$$
$$v_4=\frac{r^2\sin^2\theta\sin^2 x}{\lambda^2}x_{00}+\lambda^2Ax_{44}$$
$$v_5=\lambda^2r^2x_{11}-\lambda^2A^{-1}x_{22}-\lambda^{-2}Ax_{23}R^3{}_{212}/R^2{}_{113} \tag{14}$$
$$v_6=\lambda^2Ax_{12}+\lambda^{-2}A^{-1}x_{13}R^3{}_{002}/R^2{}_{002}$$
$$v_7=x_{01}\qquad v_8=x_{02}\qquad v_9=x_{03}$$
$$v_{10}=x_{04}\qquad v_{11}=x_{14}\qquad v_{12}=x_{24}$$
$$v_{13}=x_{34}.$$

The vector ω in X which is orthogonal to all the v_α obtained above is

$$\omega=\lambda^2Ax_{00}-\lambda^2A^{-1}x_{11}-\lambda^2r^2[x_{22}+\sin^2 x(x_{33}+\lambda^{-4}\sin^2\theta\,x_{44})]. \tag{15}$$

Equation $kg_{\mu\nu}=(\omega x_{\mu\nu})$ now gives

$$\mathrm{d}s^2=\phi\{\lambda^2A\,\mathrm{d}t^2-\lambda^2A^{-1}\,\mathrm{d}r^2-\lambda^2r^2[\mathrm{d}x^2+\sin^2 x(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\lambda^{-4})]\}. \tag{16}$$

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